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A Monte Carlo Study of the Gambler's Ruin Problem

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Abstract:
The Gambler's Ruin problem is one of the classical stochastic models of an agent who gains or loses capital repeatedly, till reaching his target level or till his ruin. In this paper, the problem is theoretically analyzed and solved by Monte Carlo simulation. This paper obtains the closed-form expression for the ruin probability and discuss its sensitivity with respect to the initial capital and the probability bias. The simulation results are close to the theoretical predictions and show how the results of the long-term are sensitive to the small changes of the probability. This paper also carries out an analysis of the real-life applications in the finance and economy field in addition to theoretical and numeric analysis. In particular, this paper looks at the applicability of the Gambler's Ruin problem in the case of grid trading strategies, asymmetric investment decisions, and firm survival dynamics. The applications in the paper show the utility of the model in the areas of long-term risk accumulation, capital sustainability, and decision making under uncertainty. The results show how critical probabilistic reasoning is to assess strategies that might seem to be profitable on the short term but have high risks on the long term.

Keywords: Gambler's Ruin; Monte Carlo Simulation; Financial Risk; Random Walk; Stochastic Processes

1. Introduction

The Gambler's Ruin problem is a very classical and well-studied problem in Probability. It is a stochastic process, where a gambler repeatedly plays in a series of independent games, losing (or winning) a certain amount of capital at every step, until he hits one of two absorbing barriers: total ruin or a predetermined capital level. Although simple in its nature, the model exposes a critical part of randomness and risk: series of little random occurrences can aggregate to an extreme event. This phenomenon is important in real world applications, such as finances and economics. In today's financial markets traders and investors are constantly subjected to strings of unpredictable events. While they may seem small in the short-term, the overall impact of these ups and downs can accumulate and become a risk over time. For instance, a strategy that makes small amounts of profit with a high frequency can still suffer from rare but large losses, and lose all its capital. In the same way, companies with uncertainty in the market have volatile

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revenues and expenses, and if the adverse market period persists, they may go bankrupt.

One framework that can be used to simplify and powerfully analyze such situations is the Gambler's Ruin framework [1,2]. The model permits to quantify the probability of ruin and how it depends on the basic parameters of the model, such as initial capital and the probability of winning, by modelling capital as a stochastic process with absorbing boundaries. A key takeaway from this framework is that even though a process might seem profitable in the short term, there could still be a high likelihood of ruin in the long term, especially when the agent has limited resources. A second noteworthy feature of the Gambler's Ruin problem is that it is related to the theory of random walk [3]. The gambler's capital can be thought of as a discrete time random walk with absorbing states. Such a relation enables people to use various mathematical tools, such as difference equations and probabilistic analysis, to obtain closed form solutions for important quantities like ruin probability, or expected stopping time [4,5]. These analytical results can be used to give useful benchmarks in the evaluation of more complicated stochastic systems.

This paper is about the Gambler's Ruin problem from a theoretical and computational point of view. This article first obtains the classical formula for the probability of ruin, by difference equation methods. Second, this paper simulates these results using Monte Carlo simulation which will give people the opportunity to see the behavior of the system in multiple trials. This paper hopes this blend will make it easier to see and understand the relationship between the probabilistic dynamics and the long-term outcomes.

The remainder of this paper is organized as follows. Section 2 is dedicated to the theoretical aspects and the main analytical results are derived. The methodology of the simulation, which was used to approximate the probability of ruin, is described in Section 3. The results are presented in section 4 and the theoretical predictions are compared with the empirical results. Finally, Section 5 concludes the paper, and suggests future research directions.

2. Theoretical Framework

This paper is considering a gambler with an initial capital i ($0 < i < N$). The gambler's capital is a stochastic process defined in a discrete time. This goes on until the gambler's fortune is reduced to one of two absorbing values: 0 (ruin) and N (success). The process stops when either of the borders is reached.

Let u_i be the probability that the gambler will eventually be ruined if he starts with capital i. Thus, conditioning on what happened at the first step, this article gets the following recurrence relation:

$$u_i = pu_{i+1} + qu_{i-1} \tag{1}$$

The fact that the probability of ruin starting from state independents on the probabilities of moving to states $i+1$ and $i-1$ is reflected in this equation. The recurrence relation's boundary conditions are $u_0 = 1, u_N = 0$. These conditions represent the situation that if the gambler starts with 0 capital, then ruin has already occurred, and if the gambler reaches the target level N, then ruin is not.

The recurrence relation is a linear difference equation of order 2 [6]. General solution is of the form:

$$u_i = A + B\left(\frac{q}{p}\right)^i \tag{2}$$

where A and B are constants which depend on the boundary conditions. Now putting the boundary conditions into the general solution, this article finds:

$$u_i = \frac{\left(q/p\right)^i - (q/p)^N}{1-(q/p)^N} \tag{3}$$

The general case (where p is not equal to q) is given by this expression.

When $p = q = 0.5$, then $q/p = 1$, and the expression becomes quite simple. Hence, the solution is:

$$u_i = 1 - \frac{i}{N} \tag{4}$$

This linear relationship indicates that the probability of bankruptcy is linear with the initial capital. That is, the longer a gambler can continue, the less likely he is to run out of capital.

The other important quantity is the expected stopping time which is the mean number of steps that it takes before the process hits one of the two absorbing boundaries. For the fair case ($p = 0.5$), the expected stopping time is:

$$E\left[T_i\right] = i\left(N-i\right) \tag{5}$$

This expression reveals that the duration of the process is expected to be the largest when the initial capital is in the middle value of the process, $i = N/2$. This is consistent with the fact that it takes longer to reach a boundary if the process is started further away from both boundaries [6]. These theoretical results give a complete description of the basic properties of the Gambler's Ruin problem. These are also used as a test case for the accuracy of simulation techniques.

3. Methodology

To complement the theoretical analysis, this paper use Monte Carlo simulation to estimate the probability of ruin and explore the behavior of the process under repeated trials. One important usage of simulation is when considering stochastic systems, such that one can see what happens instead of just trying to analyze it [6].

The simulation procedure is aimed at stimulating the random process that is described in the theoretical model. In each game, the bettor begins with an upper-case i. A random number is drawn from a uniform distribution (range 0,1) at each step. If the random number is smaller than p, the gambler gets a unit of capital, and if it is larger, the gambler loses a unit of capital.

The game is played until the gambler's capital is down to zero or to the desired level N. Each of these trials can be either ruin or success. This paper will repeat this process many times to get an empirical estimate of the probability of ruin.

Formally, if this paper carries out M independent trials and see ruin in R of them, then this paper can estimate the probability of ruin to be $\hat{u}_i = \frac{R}{M}$. The more trials that are performed, the closer the probability estimate should be to the actual probability of ruin.

The simulation not only provides an estimate of the probability of ruin but can also provide an estimate of the expected stopping time. This is done by noting the number of steps for each trial to reach an absorbing boundary, and then by taking the average of all of the trials.

A benefit of simulation is that it is flexible. The model in the paper is fairly simple, with the probabilities fixed and the step sizes fixed, but simulation can be readily generalized to more complicated situations, in which the probabilities are allowed to vary, the payoffs may be asymmetric, and there may be more constraints. But simulation is not without its drawbacks, such as the expense of running the simulation and the statistical error that accompanies the results; these must also be taken into consideration when interpreting the results of the simulation.

4. Results and Discussion

The results of the simulation are in good agreement with the theoretical calculations. For the fair case when $p = 0.5$, the estimated probability of ruin is very close to the linear relationship predicted by the analytical solution. This gives good indication of the validity of the model and simulation process.

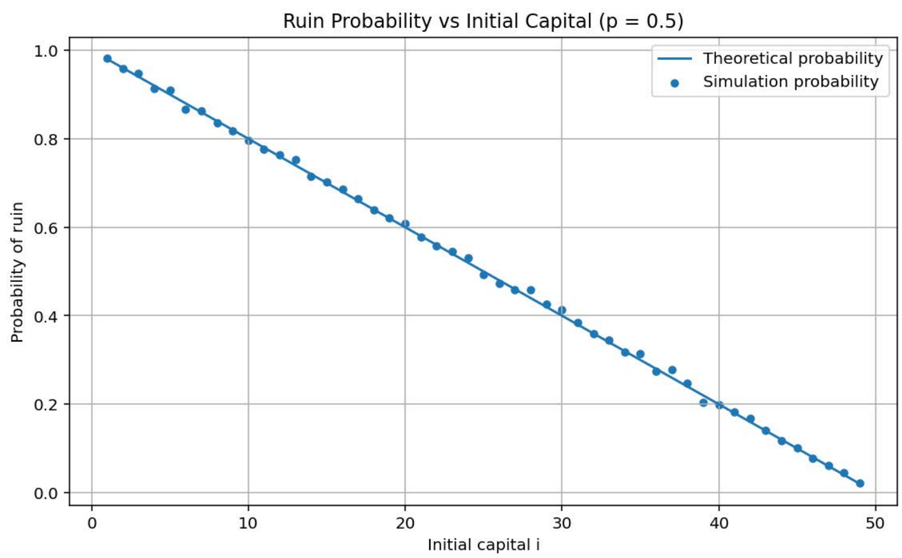

Fig. 1 Theoretical and simulated ruin probabilities versus initial capital for $p = 0.5$.

As seen in figure 1, this probability of ruin is a linear function of initial capital. Perhaps the most significant observation is that the probability of ruin is sensitive to the parameter p. If the gambler's probability of ruin is greater than 0.5, then the gambler has a disadvantage and the probability of ruin rises rapidly even for fairly large amounts of gambler's initial capital. On the other hand, if p>0.5, the gambler will have the upper hand, and the chances increase greatly that this gambler will get to the desired level.

This sensitivity brings to mind an important lesson: Probability can have a significant effect on longer-term consequences [5]. As can be seen in figure 2 even slight deviations from $p = 0.5$ result in significant differences in ruin probability.

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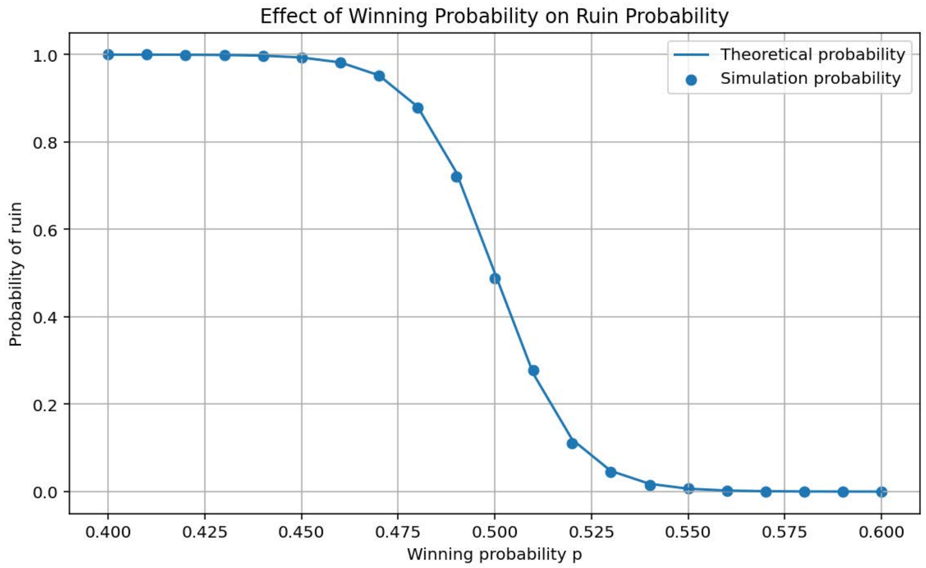

Fig. 2 Effect of winning probability p on the probability of ruin.

From the practical point of view, this means that even a small disadvantage in a process if repeated many times, can over time mean a high probability of failure. This is especially true in the financial world, where slight variations in the expected return can sum up to be quite risky.

Analysis of the expected stopping time also is informative. The results of the simulation show that the expected time of the process is maximum at the mid-point of the initial capital.

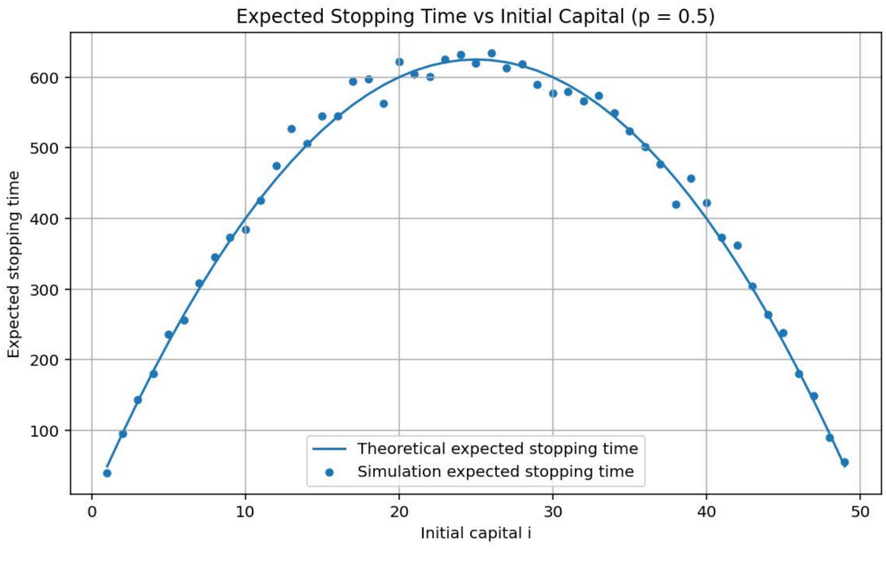

Fig. 3 Theoretical and simulated expected stopping times versus initial capital for $p = 0.5$.

The stopping times expected are quadratic as shown in Figure 3. This is because the time required to reach an absorbing boundary is greater if the process is started further away from either boundary. Another crucial point to note is the difference in results. Although the expected probability of ruin is moderate, there may be a lot of variation in the outcome of the individual trials. Trials can be ruined in just a few steps or be carried on for many steps before entering an absorbing state. This variation emphasizes the need to pay attention to not only the expected values, but also the distribution of values. The results overall indicate that the Gambler's Ruin model is a very useful and easy to understand model of the dynamics of stochastic processes. It emulates the meaningful aspects of risk accumulation and emphasizes the need for probabilistic thinking when making decisions under uncertainty.

5. Applications

5.1 Application in Grid Trading and Investment Strategies

In Gambler's Ruin Problem and Bi-directional Grid Constrained Trading and Investment Strategies, Taranto and Khan offer a straightforward financial application of the Gambler's Ruin framework. The paper investigates bi-directional grid constrained trading strategies, particularly in financial markets like the foreign exchange. Grid trading is a method that involves setting predetermined price levels for buying and selling trades, allowing the trader to take advantage of the many short-term price movements. The idea on the surface is appealing: Profiting on a regular basis from a volatile market. But, as the paper states, they're very similar to the classical Gambler's Ruin problem, as the trader could eventually run out of money in the trading account due to repeated exposure to the market movements. The authors make it clear that while grid trading can generate profits in the short term for many traders, it can virtually destroy an investment account in the long run [7].

The model investigated in this paper is well related to this application [8]. The classical Gambler's Ruin problem is a problem where a gambler starts at his initial capital and wins or loses a unit at a time repeatedly until he either hits the jackpot or goes bankrupt. Likewise, with grid trading, the investor is constantly taking trades and the profit or loss they make from these trades depends on the market. The absorbing boundary at zero is where the trader's account is no longer able to sustain any more margin calls or losses. The top ceiling is one based on a desired portfolio value or a profit level. Thus, the ruin probability obtained in the theoretical part can be regarded as an approximate measure of the probability of ruin of a trading strategy.

The significance of this application is to recognize that what is profitable in the short term does not always mean what is safe in the long-term. An individual can win a lot of trades while using a grid trading strategy, or perhaps a gambler can win lots of rounds in advance of losing everything. The crucial question is whether or not returns are averaged over the number of trades, or what is the probability of a string of losses. This is the kind of risk that the Gambler's Ruin model would cover [5]. Thus, although each loss may occur with low frequency, repeated losses may occur with high frequency making eventual ruin likely, particularly if the trader has limited capital.

In addition, Taranto and Khan depart from the classical setting to talk about semi martingales and more realistic market processes. This is crucial since financial prices don't always rise and fall on their own. The classical model is still useful, however, as a benchmark. It offers a clear mathematical explanation of what is the role of capital size, probability bias and boundary conditions. The application verifies that the result of the simulation in this paper will show that if a little change is introduced in the probability of winning the game, it can have a huge impact on the results in the long run. Risk management, therefore, should be based not only upon the expected return on the account, but also on the likelihood of the account being depleted over time, for investment strategies.

This paper as a whole is a useful application as it involves a simple probability model and a realistic financial problem. In addition to being a theoretical problem, it serves as a cautionary tale about trading strategies which might look great in the short run, but have a ruin risk hidden in the longer time frame: Gambler's Ruin.

5.2 Application in Asymmetric Payoff Investment Decisions

5.2.1 Asymmetric Payoffs and the Gambler's Ruin Extension

The Gambler's Ruin with Asymmetric Payoffs by Whelan, is an extension of the classical Gambler's Ruin problem, where gains and losses may be of differing sizes. In the "normal" version of the problem, the gambler bet wins or loses one unit per step. But many real-life financial decisions are asymmetric; either one could gamble a small sum for a big return or one could have a small return a lot of the time and a large loss sometimes. Whelan illustrates with venture capital, derivative contracts, sports betting, as well as other investments in which the chance of success is minimal, but the reward is great [9].

Whelan’s model is particularly relevant to the simulation model used in this study [10]. In this study, the simulation assumes that each step changes the gambler’s capital by exactly one unit. This assumption simplifies the analysis and allows the simulation results to be compared directly with the classical theoretical formula. However, real investment decisions are rarely symmetric in this way. In many financial settings, the amount gained after success may be different from the amount lost after failure. For instance, in many start-ups a VC might lose 100% of the investment, but if he is able to get a VC exit, he will earn a lot of money from one successful company. Likewise, an option trader might be losing some small amounts of money over and over, but if the stock takes a big leap in the right direction, he will make a significant profit. The classical model can be extended to provide greater payout for the winner than the amount that the loser wagers, as illustrated in these examples.

Whelan's results are significant because they illustrate that positive expected return does not necessarily rule out risk of ruin. This is more directly applicable with financial decision making. A strategy might have a positive expected value, but if the chances of a success are slim, and the

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investor has not a lot of money, the investor can get wiped out before the big win is realized. This idea is one of the core concepts of the Gambler's Ruin problem, namely that survival is determined by the average pay-off, capital reserves, probability structure and stopping boundary.

The paper also provides an explanation for the significance of the initial capital. The classical model implies that the higher the initial amount of capital of a gambler, the smaller is the probability of hitting the zero. In the case of an asymmetric investment situation, this still holds true, but it is more complicated. For the pay-off to be great but infrequent, the investor needs to live long enough to see the pay-off event. Hence, if people do not have enough capital in the beginning, a strategy that is initially profitable can be risky to implement. This relates directly to the results of the simulation in this paper, which showed that increasing the initial capital decreases the probability of ruin.

5.2.2 The Role of Initial Capital and Financial Risk

This application also boosts the understanding of financial risk. Traditional expected value analysis might be used to determine that a strategy is desirable, but from the Gambler's Ruin point of view, the question is whether the investor will survive the process of losses until the expected gain can be achieved. This is particularly true for leveraged investing, options trading and venture capital portfolio investments. Path dependence makes a difference in all these cases.

Thus, Whelan's paper is in terms of theory a good extension of the model. It demonstrates that the Gambler's Ruin model can be applied to simple fair or unfair games, as well as to investment issues in the modern world, in which both the risks and rewards are asymmetric. This renders the model more realistic and enhances its applicability in the field of financial decision making.

5.2.3 Application to Firm Survival and Bankruptcy

Bankruptcy risk and firm survival are of particular interest in the context of applications. Coad, Frankish, Roberts and Storey's paper shows growth paths and survival chances. An application of Gambler’s Ruin Theory uses the Gambler’s Ruin approach to survival of new firms. The paper assumes that the gambler is not only a particular player but a firm with a stock of resources as an economic agent. These resources can rise or fall depending on the elements of the business' success or failure. When the firm's resource reaches zero, the firm leaves the market, and this is like ruin in the classical model. The authors analyze 6, 247 UK start-ups for six years and suggest that firm performance can be represented as a random walk and survival is dependent upon the cumulative stock of resources [11].

This application is very pertinent as it demonstrates how the Gambler's Ruin problem can be applied to fields other than gambling and financial trading itself [6]. Every new business experiences the same type of results over and over again; sometimes sales go up, sometimes they go down; sometimes costs go up, sometimes down; sometimes customers come and sometimes they go, and sometimes markets shift. These periods may be thought of as a “win” or “loss” for the firm's resources base. If the firm amasses a number of positive periods, it might develop and live. Too many adverse shocks can lead it to run out of resources and to fail.

The linkage with the classical model is evident. The gamblers' fortune is required to hit zero or some fixed amount. For the business survival context, 0 indicates bankruptcy or dropping out of the market while the top boundary may be a stable growth or profitability or even long-term survival. The amount of money the firm has at the start is equal to the amount of money that the gambler has at the start. A company with a larger capital can withstand losses more easily than a small company, and a gambler with a bigger bankroll has a smaller chance of being ruined.

It is also a good example of path dependence. Two firms can have similar average performance, and yet have different survival prospects because of the order of gains and losses. Even if the long-term expected performance of a firm is not poor, it can experience a few losses early and not survive to capitalize on the gains it would have obtained in the long run. This is like a gambler with an eventual good chance of winning, but who goes bankrupt before he or she gets there. Hence, the model is useful in explaining the factors that make early-stage companies vulnerable and the importance of capital.

The application of firm survival is wider and more realistic than the financial trading. It demonstrates that the Gambler's Ruin model can be used to model any scenario where an agent repeatedly has to take uncertain gains and losses, but has limited resources. The model proves valuable for the analysis of bankruptcy risk, start-up failure and company resilience. It also gives a theoretical explanation why larger firms with more capital or better management of cash-flow, or low exposure to negative shocks are more likely to survive.

5.2.4 Connection to the Simulation Results

The simulation results in this paper are backed up by the firm survival application. A relationship between the probability of ruin and the amount of initial capital is illustrated in the simulation, and it is seen that relatively small changes in the probability of ruin can have a significant impact on the long-term consequences. For firms, this implies that slight improvements in profits, costs or market conditions can have large impacts on likelihood of survival over time. The Gambler's Ruin framework can therefore be seen as a simple yet powerful framework for understanding the risk of bankruptcy and business survival when uncertainty exists.

Conclusion

This paper has explored the Gambler's Ruin problem theoretically and computationally. This paper has shown the analytical solution for probability of ruin, and compared it with the Monte Carlo simulation to show the consistency between theory and empirical results. The results indicate that there are several key takeaways. The first is a strong dependence of the probability of ruin on the initial capital and on the probability of success. Secondly, large numbers of trials can cause a small difference in probabilities to make a big difference in the long-term result. Thirdly, the length of time that the process will take depends on the location of the initial capital with respect to the boundaries. The results highlight the need for probabilistic reasoning when dealing with uncertainty in a system. Simple models can yield useful information about a complex phenomenon (as long as they retain the key features of the phenomenon). It is possible to extend the model to more realistic situations in the future, for example, time varying probabilities, asymmetrical payoffs and multi-dimensional processes. Such extensions would enhance the applicability of the Gambler’s Ruin framework and provide deeper insights into real-world problems.

However, this study also has some limitations. The simulation model used in this paper is based on several simplifying assumptions. For example, the probability of success is fixed, the step size is constant, and the boundaries are clearly defined. These assumptions make the model easier to analyze, but they also mean that the model cannot fully represent complex real-world situations. In financial or business applications, probabilities may change over time, gains and losses may be asymmetric, and external factors may affect decision making. Therefore, the results should be understood as an illustration of the main principles of the Gambler’s Ruin problem rather than a complete description of all real-world risk processes.

Future research could improve this model in several directions. One possible extension is to consider time-varying probabilities, where the chance of success changes during the process. Another extension is to study asymmetric payoffs in more detail, so that each win and loss does not have the same size. It would also be useful to examine multi-dimensional models, where several risk factors interact with each other. These developments would make the Gambler’s Ruin framework more realistic and more applicable to financial investment, insurance, and business survival problems.

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